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Approximation of Parametric Surfaces

W. L. F. Degen

Mathematisches Institut B, University of Stuttgart, Vaihingen, Pfaffenwaldring 57, 2. Lehrstuhl, 70550 Stuttgart, Germany

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A general method to determine the approximation order of a triangular surface segment with planar boundary curves by a suitable Bézier patch is developed; the method is based on the affine invariance of the approximation order and uses adapted coordinate systems. As an example the case of a quadratic Bézier approximant is worked out in detail and results in the approximation order of three.

Key Words: approximation order; parametric surfaces; triangular Bézier patches.

1. INTRODUCTION

While the high accuracy approximation of parametric curves by geometric splines has attained a high level of standard (see [1, 3–5, 7, 10, 12, 17, 18]) only very few attempts have been made in the past for surfaces (see [2, 8, 9, 11, 15, 16]). Clearly, there are different schemes in use, however an error estimation or investigations on the order of approximation are missing. This problem was recently attacked by G. Mehl in her thesis [14] where she proved that an arbitrary triangular surface patch with planar boundary curves can be approximated (under mild regular conditions) by cubic triangular Bézier surfaces with the approximation order of five.

However, very tedious mappings and reparametrizations with huge calculations were necessary to obtain this result. In this paper we present a new method allowing to determine the approximation order in a very natural way for triangular Bézier approximants of any degree in principle. (Of course, the calculations grow up rapidly with that degree.) The basic idea is to observe that the order of contact as well as the approximation order (suitably defined) are invariant under affine transformations; so we can adapt the coordinate system closely to the problem. (A discussion of that invariance property can be found in [4, 5].)

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As an example, the quadratic case—not contained in [14]—is worked out in detail. As in the cubic case the approximation order is one less than that of the planar boundary curves and equals three in this case.

2. THE SIMPLEX OF REFERENCE

Throughout the paper we consider a surface $\mathbf{S}$ represented in *parametric* form by a C^∞ mapping

$$X\colon G \to \mathbb{R}^3 \tag{2.1}$$

of an open connected domain G in $\mathbb{R}^2$ (the parameters being denoted by (u, v)). From this (global) surface $\mathbf{S}$ we take a triangular segment $\mathbf{S}_h$ being the image of a certain parameter triangle Δ_h depending on a step parameter $h \in \mathbb{R}^+$; without loss of generality we can choose the parameterization in such a way that Δ_h, is defined by

$$\Delta_h := \{(u, v) \in \mathbb{R}^2 \mid u \in [0, h], v \in [0, h], u + v \leqslant h\} \tag{2.2}$$

and that $\Delta_h \subset G$ for all $0 < h < h_1$ and some $h_1 \in \mathbb{R}^+$. Thus $\mathbf{S}_h$ is well-defined for all those values of h.

In particular, the vertices of $\mathbf{S}_h$ are given by

$$A = X(0, 0), \qquad B = X(h, 0), \qquad C = X(0, h) \tag{2.3}$$

and the three boundary curves of $\mathbf{S}_h$ are the restrictions of (2.1) to the edges of Δ_h:

$$\begin{aligned} \mathscr{C}_{AB}&\colon \tilde{u} \mapsto X(u, 0), && u \in [0, h], \\ \mathscr{C}_{AC}&\colon \tilde{v} \mapsto X(0, v), && v \in [0, h], \\ \mathscr{C}_{BC}&\colon t \mapsto X(h - t, t), && t \in [0, h]. \end{aligned} \tag{2.4}$$

Furthermore, we assume that the following conditions (a)–(d) are satisfied for all $h \in (0, h_1)$:

(a) The three boundary curves $\mathscr{C}_{AB}$, $\mathscr{C}_{BC}$, $\mathscr{C}_{CA}$ are *planar* curves; their planes either intersect in one single point D different from any of A, B, C or they intersect by pairs in three distinct parallel lines.

We denote the three planes of $\mathscr{C}_{AB}$, $\mathscr{C}_{BC}$, $\mathscr{C}_{CA}$ by γ, α, β respectively and the intersection lines by $\mathbf{a} := \gamma \cap \beta$, $\mathbf{b} := \alpha \cap \gamma$, $\mathbf{c} := \beta \cap \alpha$; so by (a) these lines either have the single point D in common or they are parallel to each other. We think of the Euclidean space to be projectively extended by a plane at infinity; thus we can consider the direction of the three parallel lines as a point D at infinity. We refer to this case as the "prismatic case."

In the first case when D is a finite point, the three half-lines starting at D and containing one of the points A, B, C will be denoted by a^+, b^+, c^+ respectively. Their convex hull is a closed (conic, unbounded) part of space denoted by $\mathbf{P}$. In the prismatic case we take as $\mathbf{P}$ simply the convex hull of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, thus being a prisma unbounded at both sides. Finally, we denote the faces of $\mathbf{P}$ by α^+, β^+, γ^+ (so, for instance $\alpha^+ = conv(\mathbf{b}^+, \mathbf{c}^+)$ for finite D and $\alpha^+ = conv(\mathbf{b}, \mathbf{c})$ for the prismatic case where *conv* means the convex hull; likewise by cyclic permutation for β^+, γ^+). With these denotations we formulate the next condition as:

(b) The surface segment $\mathbf{S}_h$ is completely contained in $\mathbf{P}$ and the projection p of $\mathbf{S}_h$ (either from D or by projection rays parallel to $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in the prismatic case) onto the base triangle $\Delta(ABC)$ is a diffeomorphism (p bijective and both p and p^{-1} C^∞-differentiable).

Clearly, the projection of $\mathbf{S}_h$ onto any other profile triangle will also be a diffeomorphism because the projection of two such triangles has this property and diffeomorphisms can be composed. (Later we will use a triangle cut out of $\mathbf{P}$ by the tangent plane of $\mathbf{S}_h$ at $\mathbf{A}$.)

Condition (b) implies that neither the surface $\mathbf{S}_h$ nor any tangent plane $T_X\mathbf{S}_h$ (at X, tangent to $\mathbf{S}_h$) contains D (nor is parallel to $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ in the prismatic case) because the projection p would be singular at X. In particular, the three tangent planes $T_A\mathbf{S}_h$, $T_B\mathbf{S}_h$, $T_C\mathbf{S}_h$ do not contain the lines $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, respectively. Thus they are *transversal* to $\mathbf{P}$ and consequently they intersect the neighboring faces of $\mathbf{P}$ in well-defined lines

$$\begin{aligned} \mathbf{a}_1 &:= T_A\mathbf{S}_h \cap \gamma, & \mathbf{a}_2 &:= T_A\mathbf{S}_h \cap \beta, \\ \mathbf{b}_1 &:= T_B\mathbf{S}_h \cap \alpha, & \mathbf{b}_2 &:= T_B\mathbf{S}_h \cap \gamma, \\ \mathbf{c}_1 &:= T_C\mathbf{S}_h \cap \beta, & \mathbf{c}_2 &:= T_C\mathbf{S}_h \cap \alpha. \end{aligned} \tag{2.5}$$

later, in Section 4, when dealing with quadratic Bézier approximants, we shall need still two more conditions in order to guarantee their existence, but they are not needed for the general approach:

(c) The three pairs of lines from (2.5) lying in the same face intersect in an *interior* point of that face:

$$F := \mathbf{a}_1 \cap \mathbf{b}_2 \in \gamma^+, \qquad G := \mathbf{b}_1 \cap \mathbf{c}_2 \in \alpha^+, \qquad E^+ := \mathbf{c}_1 \cap \mathbf{a}_2 \in \beta^+. \tag{2.6}$$

(d) The boundary curves $\mathscr{C}_{AB}$, $\mathscr{C}_{BC}$, $\mathscr{C}_{CA}$ have non-vanishing curvature at all of their points.

These conditions seem to be rather restrictive. However, in reality, they are partly redundant and may be weakened. For example, they are fulfilled for any sufficiently small triangular patch on a surface with positive

Gaussian curvature. We renounce the lengthy discussion of all possibilities to reduce them in other cases and take them for simplicity as they are.

We wan to approximate $\mathbf{S}_h$ by a triangular Bézier surface $\mathbf{A}$ represented as usual by

$$\mathbf{A} \cdots Y(\mathbf{u}) = \sum_{|I|=n} \frac{n!}{I!} \mathbf{u}^I C_I \tag{2.7}$$

using multi-index notation ($\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{I} = (i_1, i_2, i_3)$ $|\mathbf{I}| = i_1 + i_2 + i_3$, $\mathbf{u}^{\mathbf{I}} = u_1^{i_1} u_2^{i_2} u_3^{i_3}$, $\mathbf{I}! = i_1!\, i_2!\, i_3!$) and barycentric coordinates u_1, u_2, u_3 satisfying

$$u_1 + u_2 + u_3 = 1. \tag{2.8}$$

Excluding the trivial case $n = 1$ where $\mathbf{A}$ is simply the base triangular itself, we require for $\mathbf{A}$:

(e) The approximant $\mathbf{A}$ is a regular triangular Bézier surface of degree $n \geqslant 2$, having the same simplex (resp. prism) of reference as $\mathbf{S}_h$ (planer boundary curves in planes α, β, γ respectively and $\mathbf{A}$ completely contained in $\mathbf{P}$) and (like $\mathbf{S}_h$) having the property that the projection $p_{\mathbf{A}}$ from D (the parallel projection in the prismatic case) onto $\Delta(ABC)$ is a diffeomorphism.

(f) $\mathbf{A}$ is tangent to $\mathbf{S}_h$ at all the three vertices A, B, C.

Most of these properties can be expressed in terms of the control points; Since A, B, C are the vertices also of $\mathbf{A}$, we have

$$A = C_{n,0,0}, \qquad B = C_{0,n,0}, \qquad C = C_{0,0,n}. \tag{2.9}$$

Since the boundary curves must be, by (e), in α^+, β^+, γ^+, so are their control points:

$$\left.\begin{aligned} C_{n-k,k,0} &\in \gamma^+ \backslash D \\ C_{n-k,0,k} &\in \beta^+ \backslash D \\ C_{0,n-k,k} &\in \alpha^+ \backslash D \end{aligned}\right\} \qquad \text{for all} \quad k \in 0 \cdots n. \tag{2.10}$$

Finally, since $\mathbf{A}$ is tangent to $\mathbf{S}_h$ at A, B, C by (f) we have (see (2.5)):

$$\begin{aligned} &C_{n-1,1,0} \in \mathbf{a}_1 \cap \gamma^0, && C_{n-1,0,1} \in \mathbf{a}_2 \cap \beta^0 \\ &C_{0,n-1,1} \in \mathbf{b}_1 \cap \alpha^0, && C_{1,n-1,0} \in \mathbf{b}_2 \cap \gamma^0 \\ &C_{1,0,n-1} \in \mathbf{c}_1 \cap \beta^0, && C_{0,1,n-1} \in \mathbf{c}_2 \cap \alpha^0, \end{aligned} \tag{2.11}$$

where α^0, β^0, γ^0 denote the interior parts of α^+, β^+, γ^+, respectively.

Since a Bézier curve of surface is contained in the convex hull of its control points the conditions (2.9)–(2.11) are almost sufficient for (e), (f): Only the requirement that the projection $p_{\mathbf{A}}$ is a diffeomorphism is not yet expressed by them.

In order to give a precise definition of what is meant by "approximation order" we start with barycentric coordinates (x_1, x_2, x_3) with respect to the basic triangle $\Delta(ABC)$ and extend them to coordinates in space. In the case of a finite simplex $ABCD$ we may use barycentric coordinates $(\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{x}_4)$ with respect to $ABCD$ (related by $(\underline{x}_1+\underline{x}_2+\underline{x}_3+\underline{x}_4=1)$) and then obtain for any point X in space—different from D—its central projection X' (from D onto the plane ABC) by

$$x_i = \frac{\underline{x}_i}{1-\underline{x}_4} \qquad (i=1, 2, 3). \tag{2.12}$$

Instead of $\underline{x}_4$ we use

$$z = \frac{\underline{x}_4}{1-\underline{x}_4} \tag{2.13}$$

as a fourth coordinate determining X in space when its projection X' in ABC is given. From elementary geometry we take, that (2.12), (2.13) describe a projective transformation of space leaving invariant the basic triangle $\Delta(ABC)$ and bringing D to infinity. *Hence the case of a simplex ABCD is reduced to the case of a prism and z is an affine coordinate along its spatial direction.*

We think this transformation to be carried out in every case where a simplex occurs and so we can confine us to the case of a prism using parallel projections of the surface segment $\mathbf{S}_h$ and its approximant $\mathbf{A}$ onto the basic triangle $\Delta(ABC)$.

DEFINITION. Let denote X' an arbitrary point in $\Delta(ABC)$ and $z_{\mathbf{S},h}$, $z_{\mathbf{A}}$ the z-coordinates of the preimages $X \in \mathbf{S}_h$ and $Y \in \mathbf{A}$ of X' by parallel projection in the direction of the z-axis, then the greatest number $k \in \mathbb{N}$ such that

$$\|z_{\mathbf{S},h} - z_{\mathbf{A}}\| = \mathcal{O}(h^k) \tag{2.14}$$

holds for all $X' \in \Delta(A, B, C)$ is called the *approximation order* of $\mathbf{A}$ to $\mathbf{S}_h$.

Remarks. (1) This definition of the approximation order is by construction *affine invariant*: Note that affine transformation induce linear transformations $z' = az + b$ thus $|z'_{\mathbf{S},h} - z'_{\mathbf{A}}| = |a| \cdot |z_{\mathbf{S},h} - z_{\mathbf{A}}|$ proving that k is invariant.

(2) We had to change the notation between the two kinds of barycentric coordinates: $\mathbf{u} = (u_1, u_2, u_3)$ denotes a point in the parameter triangle for $\mathbf{S}_h$ and $\mathbf{x} = (x_1, x_2, x_3)$ are those of X' in $\Delta(A, B, C)$. Though we have a bijective mapping $\phi: \mathbf{u} \to \mathbf{x}$, ϕ may not be linear and thus $\mathbf{u} \neq \mathbf{x}$ (see the next section).

(3) The transition from $\underline{x}_4$ to z by (2.13) does not influence the approximation order, since we may expand (2.13) as a geometric series (for h small enough to fulfill $|\underline{x}_4| < 1$) obtaining

$$|z_{\mathbf{S}_h} - z_{\mathbf{A}}| = |\underline{x}_{4,\mathbf{S}_h} - \underline{x}_{4,\mathbf{A}}| \cdot |1 + \mathcal{O}(h)|.$$

3. DETERMINING THE APPROXIMATION ORDER

Now we describe a general approach to determine the approximation order (see Section 2). Of course, for $n > 2$, we have to impose further geometric conditions (beyond (e) and (f)) on the approximant in order to determine it uniquely and to make that order as high as possible. This must be done for each value of n separately. So we will establish only the main idea of the procedure in principle and work out the first non-trivial case $n = 2$ in the next section.

Clearly, to determine the approximation order, one has to use a suitable Taylor expansion for $\mathbf{S}_h$, but what should be the parameters and what the expansion point? For the latter, one could take—for reasons of symmetry—the center of gravity $M = 1/3(A + B + C)$. However, the calculations become slightly simpler by using A instead of M and the tangent plane $T_A\mathbf{S}_h$ of $\mathbf{S}_h$ instead of ABC. Projecting $\Delta(ABC)$ down onto $T_A\mathbf{S}_h$ we obtain $A' = A$ and two other points B', C'; then the barycentric coordinates of $X \in \Delta(ABC)$ and its projection $X' \in \Delta(A'B'C')$ *are the same*. Now, by an affine transformation in the parameter plane we may assume without loss of generality *that the parameter triangle* Δ_h (introduced in the previous section) *coincides with* $\Delta(A'B'C')$. This means that we have an affine coordinate system with origin $A = A'$, AB' as x-axis, AC' as y-axis and the z-axis parallel to the edges of the prism as before; i.e.,

$$u = x, \qquad v = y \tag{3.1}$$

and—identifying points with their coordinates—

$$A = (0, 0, 0)^T, \qquad B' = (h, 0, 0)^T, \qquad C' = (0, h, 0)^T. \tag{3.2}$$

With this in mind, the representation (2.1) of $\mathbf{S}_h$ carries over to

$$X = X(x, y) = \begin{pmatrix} x \\ y \\ f(x, y) \end{pmatrix} \tag{3.3}$$

and the domain may be restricted to $|x| < h_1$, $|y| < h_1$. Thus we end up simply at Euler's representation

$$\mathbf{S}_h \cdots z_{\mathbf{S},h} = f(x, y), \qquad (x, y) \in \varDelta_h \tag{3.4}$$

written *in affine coordinates* (x, y, z).

Note that $z = 0$ now represents $T_A\mathbf{S}_h$ and that the transition from ABC to $T_A\mathbf{S}_h$ implies a linear transformation on z which does not influence the approximation order. The tangency of $T_A\mathbf{S}_h$ implies

$$f(0, 0) = 0, \qquad f_x(0, 0) = 0, \qquad f_y(0, 0) = 0 \tag{3.5}$$

and the asymptotic Taylor expansion of $f(x, y)$ may be written as

$$f(x, y) = \sum_{m=2}^{M} \frac{1}{m!} \left(\sum_{k=0}^{m} \binom{m}{k} a_{mk} x^{m-k} y^k \right) + \mathcal{O}(h^{M+1}) \tag{3.6}$$

with

$$a_{mk} = \left. \frac{\partial^m f}{\partial^{m-k} x \, \partial^k y} \right|_{(0, 0)}. \tag{3.7}$$

Before inserting this in (3.4) and (2.14), we introduce *normalized tangent parameters* (ξ, η) related to (x, y) by

$$x = h\xi, \qquad y = h\eta. \tag{3.8}$$

These are *independent* on h and have the domain

$$\varDelta \cdots 0 \leqslant \xi \leqslant 1, \qquad 0 \leqslant \eta \leqslant 1, \qquad \xi + \eta \leqslant 1. \tag{3.9}$$

Of course, these tangent parameters are closely related to the barycentric coordinates introduced by (2.13); one obtains at once

$$x_1 = \xi, \qquad x_2 = \eta, \qquad x_3 = 1 - (\xi + \eta). \tag{3.10}$$

Inserting (3.8) into (3.6), Eq. (3.4) finally leads to

$$z_{\mathbf{S},h} = \sum_{m=2}^{M} \frac{h^m}{m!} \left(\sum_{k=0}^{m} \binom{m}{k} a_{mk} \xi^{m-k} \eta^k \right) + \mathcal{O}(h^{M+1}) \tag{3.11}$$

whereby the expression within the parentheses is *independent* of h. Thus $z_{\mathbf{S},h}$ is known once the coefficients a_{mk} have been calculated by (3.7).

Since by (f) the approximant $\mathbf{A}$ is also tangent to Δ_h at the point A, there is a representation for $Y(\mathbf{u})$ analogous to (3.3) for $\mathbf{S_h}$. From the former we take the quantity $z_{\mathbf{A}}$ needed in (2.14) to determine the approximation order.

For this purpose we firstly replace the barycentric coordinates (u_1, u_2, u_3) by

$$u_1 = 1 - s - t, \qquad u_2 = s, \qquad u_3 = t \tag{3.12}$$

with $(s, t) \in \Delta$, analogous to (3.9), (3.10). Then we rewrite (2.7) as

$$\hat{Y}(s, t) = \sum_{m=1}^{n} \binom{n}{m} (1 - s - t)^{n-m} \left(\sum_{k=0}^{m} \binom{m}{k} s^{m-k} t^k C_{m-n, m-k, k} \right), \tag{3.13}$$

where the hat on Y indicates the change (3.12) of parameters.

Note that the summation in (3.13) starts with $m = 1$ since by (2.7) $A = C_{n, 0, 0}$ is now at the origin and we think of (3.13) written in coordinates. Since the parameters s, t correspond to the *normalized* coordinates we refer (3.13) also to them, i.e., (ξ, η, z). From the conditions (e) and (f) we take at the moment only the properties of the two boundary curves starting from A. They are defined by $t = 0$ and $s = 0$, lie in the planes γ with equation $\eta = 0$ and β with equation $\xi = 0$ respectively and they are tangent to the x-axis respectively to the y-axis at the origin A. Thus we obtain for their control points next to A

$$C_{n-1, 1, 0} = \begin{pmatrix} \frac{1}{n} b \\ 0 \\ 0 \end{pmatrix}, \qquad C_{n-1, 0, 1} = \begin{pmatrix} 0 \\ \frac{1}{n} c \\ 0 \end{pmatrix} \tag{3.14}$$

with some constants

$$b > 0, \qquad c > 0. \tag{3.15}$$

(These constants do not vanish because otherwise $\mathbf{A}$ would be singular; they are positive because the corresponding boundary curves of $\mathbf{S}_h$ and $\mathbf{A}$ start from A in the same direction; the factors $1/n$ are added only for convenience.) Thinking (3.13) to be expanded by powers of s and t, there is no absolute term and the linear one is, by (3.14), obtained as $(bs, ct, 0)^T$.

Thus, the first two coordinates of (3.13) read as

$$\begin{aligned} \xi &= bs + \cdots \\ \eta &= ct + \cdots, \end{aligned} \tag{3.16}$$

where the symbol $+\cdots$ indicates terms of degrees >2 of s and t together. The third coordinate of (3.13) is the desired function $z_{\mathbf{A}}(s,t)$. Equations (3.16) reflect the following important property:

LEMMA 1. *The transformation* (3.16) *is* (*in its complete form*) *a diffeomorphism from Δ onto itself.*

Proof. (ξ,η) are the normalized coordinates of the projection of $\hat{Y}(s,t)$ onto the triangle $\Delta(AB'C')$; since this is the projection $p_{\mathbf{A}}$ followed by an affine transformation and $p_{\mathbf{A}}$ being a diffeomorphism by assumption (e), the assertion follows. ∎

This lemma is the crucial point of our method: It allows to use the parameters (s,t) instead of (ξ,η) also for the surface $\mathbf{S}_h$. (Alternatively we could use (ξ,η) as common parameters for both surfaces; but then we had to *solve* Eqs. (3.16) for ξ and η instead simply to insert them.)

Now the procedure to determine the approximation order is clear: Taking the parameters (s,t) also for the surface $\mathbf{S}_h$ means to insert Eqs. (3.16) into the representation

$$\hat{X}(\xi,\eta)=\begin{pmatrix}\xi\\ \eta\\ z_{\mathbf{S},h}(\xi,\eta)\end{pmatrix} \tag{3.17}$$

of $\mathbf{S}_h$ obtained by (3.3) (using a hat on X indicates the change of coordinates (3.8)). On the other hand we have an analogous representation

$$\hat{Y}(s,t)=\begin{pmatrix}\xi\\ \eta\\ z_{\mathbf{A}}(s,t)\end{pmatrix} \tag{3.18}$$

for the approximant $\mathbf{A}$ coming from (3.13) as described above. Thus—considering (s,t) (as well as (ξ,η)) as parameters for the common projection $\hat{X}'$ of $\hat{X}(\xi,\eta)$ and $\hat{Y}(s,t)$ onto the plane $AB'C'$—we can compare the two z-coordinates in (3.17) and (3.18) to get the approximation order according to its definition (2.14). However, before being able to do so, we have to compute the surface $\mathbf{S}_h$ as well as all the control points of $\mathbf{A}$ as Taylor series expansions of h up to a suitable (high enough) order and to insert these expansions into the formulas (3.17), (3.18).

As already mentioned, we do not pursue this method to the very end for a general value of n.

4. THE QUADRATIC CASE

We apply our method to determine the approximation order of the quadratic case $n=2$. First, we recall that we made the two additional assumptions (c), (d) in Section 2; then we want to point to the fact that the approximant $\mathbf{A}$ is *not a quadric*, in general, but an algebraic surface of fourth order (in general a Romanian surface of Steiner, see [6]). it has three more control points (besides the vertices A, B, C) which belong to the boundary curves one additional control point for each of them. Since these boundary curves are quadratic Bézier curves now, the intermediate control point is the intersection of the two tangents at the endpoints. By (c) and (d) these tangents are different and intersect exactly at the points E, F, G defined by (2.6) and lying in the interior of the corresponding faces α^+, β^+, γ^+ of P, respectively (see Section 2). Thus we can rewrite (3.13) as

$$\mathbf{A} \cdots Y(s,t) = 2(1-s-t)(sG+tF) + s^2B + 2stE + t^2C. \tag{4.1}$$

With respect to the original coordinate system (based on the triangle $\Delta_h = \Delta(A, B', C')$ and described at the beginning of Section 3) we have according to (2.3), (3.2), (3.3)

$$b = \begin{pmatrix} h \\ 0 \\ f(h,0) \end{pmatrix}, \quad B' = \begin{pmatrix} h \\ 0 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ h \\ f(0,h) \end{pmatrix}, \quad C' = \begin{pmatrix} 0 \\ h \\ 0 \end{pmatrix}, \tag{4.2}$$

and by (3.14)

$$G = \begin{pmatrix} \frac{1}{2}b(h) \\ 0 \\ 0 \end{pmatrix}, \qquad F = \begin{pmatrix} 0 \\ \frac{1}{2}c(h) \\ 0 \end{pmatrix}. \tag{4.3}$$

(In contrast to (3.14) we indicated the dependency on h by the notations $b(h)$ and $c(h)$ instead of b and c.) the last additional control point E lies in the plane α (face α^+), now represented by coordinates as $x+y=h$ $(0<x<h,\ 0<y<h,\ z>0)$; thus we can write

$$E = \begin{pmatrix} h/2 + a(h) \\ h/2 - a(h) \\ r(h) \end{pmatrix} \tag{4.4}$$

with suitable functions $a(h)$, $r(h)$ (restricted to $|a(h)| < h/2$, $r(h) > 0$).

Since all control points are uniquely defined by the requirements (e), (f) and being well-defined by the assumptions (a)–(d) on $\mathbf{S}_h$ we may state:

LEMMA 2. *In the quadratic case* $n=2$ *the approximant* $\mathbf{A}$ *exists and is uniquely defined by the requirements* (e), (f) *provided that* $\mathbf{S}_h$ *satisfies the assumptions* (a)–(d).

So far we have taken into account only the principal behavior of the control points B, C, E, F, G. But using the asymptotic expansion (3.11) of $\mathbf{S}_h$ that we rewrite as

$$\begin{aligned} z_{\mathbf{S},h} &= f(h\xi, h\eta) \\ &= \frac{1}{2}h^2(e\xi^2 + 2f\xi\eta + g\eta^2) \\ &\quad + \frac{1}{6}h^3(k\xi^3 + 3l\xi^2\eta + 3m\xi\eta^2 + n\eta^3) + O(h^4) \end{aligned} \tag{4.5}$$

(with constants e, f, g, k, l, m, n instead of $\alpha_{m,k}$) we can compute those points explicitly getting

$$\left.\begin{aligned} b(h) &= h + \frac{\kappa}{6}h^2 + \mathcal{O}(h^3) \\ c(h) &= h + \frac{\nu}{6}h^2 + \mathcal{O}(h^3) \\ a(h) &= \frac{1}{2}h^2\mu + \mathcal{O}(h^3) \\ r(h) &= \frac{1}{2}fh^2 + \frac{1}{12}\tau h^3 + \mathcal{O}(h^4) \end{aligned}\right\} \tag{4.6}$$

with

$$\begin{aligned} &\kappa = \frac{k}{e}, \qquad \nu = \frac{n}{g}, \qquad \mu = \frac{(1/2)(l-m) - (1/6)(k-n)}{2f-e-g}, \\ &\tau = \frac{(k-3l)(g-f) + (n-3m)(e-f)}{2f-e-g}. \end{aligned} \tag{4.7}$$

To obtain $b(h)$, say, one has to intersect the tangent

$$B(h) + s\,\frac{dB(h)}{dh} = \begin{pmatrix} h \\ 0 \\ f(h,0) \end{pmatrix} + s\begin{pmatrix} 1 \\ 0 \\ f_x(h,0) \end{pmatrix}$$

with $z=0$ getting $s=-f/f_x$; hence

$$\frac{1}{2}b(h)=h-\frac{f(h,0)}{f_x(h,0)}=h-\frac{1}{2}h\frac{(1+(1/2)\,\kappa h+\mathcal{O}(h^2))}{(1+(1/2)\,\kappa h+\mathcal{O}(h^2))}$$
$$=h+\frac{\kappa}{6}h^2+\mathcal{O}(h^3).$$

Analogously we get $c(h)$. Note that $e=f_{xx}(0,0)\neq 0$ and $g=f_{yy}(0,0)\neq 0$ by assumption (d).

The computation of E is somewhat more complicated since we have to intersect the two tangents of the third boundary curve $\mathscr{C}_{BC}$ (2.4) at $t_1=0$ and $t_2=h$

$$E=X(h,0)+s_1\left(-\frac{\partial X(h,0)}{\partial x}+\frac{\partial X(h,0)}{\partial y}\right)$$
$$=X(h,0)+s_2\left(-\frac{\partial X(h,0)}{\partial x}+\frac{\partial X(h,0)}{\partial y}\right).$$

By (3.3) this is equivalent to

$$E=\begin{pmatrix} h\\ 0\\ f(h,0)\end{pmatrix}+s_1\begin{pmatrix} -1\\ 1\\ f_y(h,0)-f_x(h,0)\end{pmatrix}$$
$$=\begin{pmatrix} 0\\ h\\ f(0,h)\end{pmatrix}+s_2\begin{pmatrix} -1\\ 1\\ f_y(h,0)-f_x(h,0)\end{pmatrix}. \tag{4.8}$$

With the denotation $Df=f_y-f_x$ we obtain the solution

$$s_1=\frac{f(0,h)-f(h,0)-hDf(h,0)}{Df(h,0)-Df(0,h)},$$
$$s_2=\frac{f(0,h)-f(h,0)-hDf(0,h)}{Df(h,0)-Df(0,h)}. \tag{4.9}$$

Note that $Df(h-t,t)$ is the slope of the boundary curve $\mathscr{C}_{BC}$; thus the denominator is the difference of the slopes at both endpoints, and this difference does not vanish by assumption (d).

Inserting (4.9) into (4.8) and performing all necessary expansions using (4.5) yields at the very end the formulas for $a(h)$ and $r(h)$ contained in (4.6); the common denominator $2f-e-g$ turns out to be equal to

$(d/dh)(Df(h,0)-Df(0,h))$ what is essentially a mean value of the curvature of $\mathscr{C}_{BC}$, hence $2f-e-g$ does not vanish by (d).

Now we are able to insert the obtained expansions of all control points into (4.1); then we pass to the normalized coordinates ξ, η instead of x, y (realizing that, indeed, one power of h cancels) and obtain after some more calculations

$$\begin{aligned} \xi &= s + h\left(\frac{\kappa}{6}(s-s^2) + st\mu\right) + \mathcal{O}(h^2) \\ \eta &= t + h\left(\frac{\nu}{6}(t-t^2) - st\mu\right) + \mathcal{O}(h^2) \end{aligned} \tag{4.10}$$

$$\begin{aligned} z_{\mathbf{A}}(s,t) = {} & \frac{1}{2}h^2(es^2 + 2stf + gt^2) \\ & + \frac{1}{6}h^3(ks^2 + \tau st + nt^2) + \mathcal{O}(h^4). \end{aligned} \tag{4.11}$$

We arrived at the stage described in Section 3 with (3.19) and consequently can proceed as explained thereafter: We use the parameters s, t also for $z_{\mathbf{S},h}$ by inserting (4.10) into (4.5) getting

$$\begin{aligned} z_{\mathbf{S},h} = {} & \frac{1}{2}h^2(es^2 + 2fst + gt^2) \\ & + \frac{1}{6}h^3[(ks^2 + (\kappa+\nu)fst + nt^2) + s^2t(6\mu(e-f) + (3l - f\kappa)) \\ & + st^2(-6\mu(g-f) + (3m - f\nu))] + \mathcal{O}(h^4). \end{aligned} \tag{4.12}$$

Comparing this expression with (4.9) shows that the quadratic terms cancel, but—in general—the cubic ones do not. The difference can be seen to be

$$z_{\mathbf{S},h} - z_{\mathbf{A}} = \frac{1}{6}h^3 st(s+t-1)[(\kappa+\nu)f - \tau] + \mathcal{O}(h^4). \tag{4.13}$$

Thus we obtained the final result:

THEOREM. *Provided the surface to be approximated satisfies the conditions* (a)–(d) *then the quadratic triangular Bézier surface* **A** *defined by the conditions* (e), (f) (*see Lemma* 2) *approximates* $\mathbf{S}_h$ *with the* third *order*.

Beyond this result Eq. (4.13) is very interesting. Since the coefficient at h^3 contains the product $st(s+t-1)$ it vanishes along each boundary curve and we may state the

COROLLARY. *Along the boundary curves the approximation order is four.*

This is not so much astonishing since it is well known that the approximation order for planar curves approximated by quadratic Bézier curves in Hermite manner is four. So our approximant **A** fails to be also of order four only by one coefficient $(\kappa+\nu)f-\tau$ that does not vanish, in general.

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